Reduced Tangent Cones and Conductor at Multiplanar Isolated Singularities

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Abstract

Let A be a noetherian local k-algebra with residue field k and associated graded ring $\mathrm{G}(A)$. Let $\mathfrak{b}$ be the conductor of A in its normalization $\overline{A}$, which we assume to be regular and finite over A. Assume also that $\mathrm{Proj}(\mathrm{G}(A))$ is reduced and $\sqrt{\mathfrak{b}} = \mathfrak{m}$. Let r be the embedding dimension of A and $H\left(\overline{A}, n\right)$ be the Hilbert function of $\overline{A}$. We show that if the Hilbert function of $\mathrm{Proj}(\mathrm{G}(A))$ is equal to $\min\left\{ \binom{n+r}{r}, H\left(\overline{A}, n\right) \right\}$ then $\mathrm{G}(A)$ is reduced. This allows us to compute, in the general case, the conductor of the local ring of singular isolated points on surfaces whose tangent cone is multiplanar; that is, it consists — as a set — of planes. We explicitly construct a class of such surfaces in the rational case.

Key Words: Associated graded ring; Tangent cone; Conductor; Singularities.

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1 Introduction

Let A be the local ring of a multiple point on a surface V, over an algebraically closed field k of characteristic zero. Assume that the tangent cone $\mathrm{Spec}(\mathrm{G}(A))$ is multiplanar (or equivalently the projectivized tangent cone $\mathrm{Proj}(\mathrm{G}(A))$ consists of lines). In (Orecchia, 2001) the conductor of A was computed under the following four assumptions: 1) the normalization $\overline{A}$ of A is regular, 2) the radical of the conductor of A in $\overline{A}$ is maximal, 3) $\mathrm{Spec}(\mathrm{G}(A))$ is reduced, 4) the planes which form the tangent cone are in generic position (that $\mathrm{Proj}(\mathrm{G}(A))$ consists of lines in generic position).

Only one example of an irreducible surface satisfying the previous hypotheses was given in that paper since condition 3) is algebraic and hard to prove.

In this paper we show that condition 3) is unnecessary. This is a consequence of the following general result.

Let A be a reduced, noetherian local k-algebra, with residue field k and embedding dimension r. Assume that the normalization $\overline{A}$ is regular and finite over A and that the radical of the conductor of A in $\overline{A}$ is maximal. Let $H(Y,n)$ and $H\left(\overline{A},n\right)$ be respectively the Hilbert function of $\mathrm{Proj}\,(\mathrm{G}\,(A))$ and of $\overline{A}$. We show that if $H(Y,n)=\min\left\{\binom{n+r}{r},\, H\left(\overline{A},n\right)\right\}$ then $\mathrm{G}\,(A)$ is reduced. In the case of multiplanar singularities on surfaces, this last conditon is equivalent to saying that $\mathrm{Proj}\,(\mathrm{G}\,(A))$ consists of lines in generic position. This last condition is very easy to prove since most of the finite sets of lines are in generic position (for example two skew lines are always in generic position). Then, to get examples of irreducible surfaces satisfying the previous conditions 1),...,4) one should construct a surface whose projective tangent cone at a point consists of given lines.

In (Geramita and Orecchia, 1982) it was shown how, when any set of projective points $\{P_1,...,P_n\}$ is fixed, it is possible to write the parametric representation of a rational curve C whose local ring A at a singular point has a projective tangent cone $\mathrm{Proj}\,(\mathrm{G}\,(A))=\{P_1,...,P_n\}$. $\mathrm{Spec}\,(\mathrm{G}\,(A))$ is not always reduced (see De Paris, 1999) but it is if the points are in generic position (see Geramita and Orecchia, 1982).

In the last part of the paper, we show an analogous construction for parametric surfaces. In fact for any set of skew lines $l_1,\ldots,l_e$ in $\mathbb{P}^r_k=\mathrm{Proj}\,(k[X_0,\ldots,X_r])$ we construct a parametric surface whose local ring A at the origin has $\mathrm{Proj}\,(\mathrm{G}\,(A))=\{l_1,\ldots,l_e\}$ and satisfies the previous conditions 1), 2). If $\{l_1,\ldots,l_e\}$ are in generic position then $\mathrm{G}\,(A)$ is reduced and it is possible to compute the conductor of A.

2 Preliminary Results

The underlying notation and results tacitly assumed in this paper can be found in (Atiyah and Macdonald, 1969) and (Hartshorne, 1977). Accordingly, all rings are assumed to be commutative and with identity element.

We fix an algebraically closed field k, and by a *surface* we mean a reduced quasi-projective scheme over k of pure dimension two. The *normalization* of a ring A will be the integral closure of A in the *total ring of fractions* $S^{-1}A$, with S being the set of all elements A that are not zero-divisors. The ring A will be said to be *reduced* if $\mathrm{Spec}\,(A)$ is so (that is, A has no nonzero nilpotent elements). The ring A will be said to be *regular* if $\mathrm{Spec}\,(A)$ is so (that is, all localization of A with respect to prime ideals are regular local rings).

If $\mathfrak{a}$ is an ideal in the ring A, then

$$\mathrm{G}_{\mathfrak{a}}\,(A)$$

will denote the associated graded ring

$$\bigoplus_{n\geq 0}\frac{\mathfrak{a}^n}{\mathfrak{a}^{n+1}}\ ;$$

if A is semilocal, the subscript will be often omitted when $\mathfrak{a}$ is the Jacobson radical (in particular, this happens when A is local and $\mathfrak{a}$ is its maximal ideal).

By the *Hilbert function* of a graded ring $G = \bigoplus_{n\geq 0} G_n$ we mean the function that with each n associates the length of the G_0–module G_n. When $G = \mathrm{G}_{\mathfrak{a}}(A)$, the value at n of the Hilbert function will be denoted by $H_{\mathfrak{a}}(A,n)$; when A is semilocal and $\mathfrak{a}$ is the Jacobson radical, we shall write simply $H(A,n)$. The Hilbert function of a closed subscheme $Y \subseteq \mathbb{P}^r_k$ will be the Hilbert function of the homogeneous coordinate ring $k[X_0,\ldots,X_r]/I(Y)$, with $I(Y)$ being the saturated ideal of Y; its value at n will be denoted by $H(Y,n)$.

The *tangent cone* of a noetherian local ring A is $\mathrm{Spec}\,(\mathrm{G}\,(A))$. The *embedding dimension* of A is defined as

$$\mathrm{emdim}(A) := H(A,1) \ ;$$

in other words, it is the dimension of the Zariski cotangent space $T^\vee := \mathfrak{m}/\mathfrak{m}^2$ at the maximal ideal $\mathfrak{m} \in \mathrm{Spec}\,(A)$ (over the residue field $A/\mathfrak{m}$). The *projectivized tangent cone* $\mathrm{Proj}\,(\mathrm{G}\,(A))$ is naturally embedded in the space $\mathrm{Proj}\,(S\,(T^\vee))$. When A is a k-algebra with residue field k, we shall tacitly fix coordinates in $\mathrm{Proj}\,(S\,(T^\vee))$, and then identify it with the projective space $\mathbb{P}^{r+1} := \mathrm{Proj}\,(k[X_1,\ldots,X_k])$ $(r+1 = \mathrm{emdim}(A))$. Accordingly, the projectivized tangent cone will be often considered as a subscheme of $\mathbb{P}^{r+1}$. Its degree coincides with the *multiplicity* $e(A)$ of the local k-algebra A.

From now on in this section, A will denote a reduced, noetherian local ring, and $\mathfrak{m}$ will be its maximal ideal. We also assume that the normalization $\overline{A}$ is finite over A and regular, and we set $\overline{\mathfrak{m}} = \mathfrak{m}\overline{A}$. By the finiteness assumption, $\overline{A}$ possesses a finite number of maximal ideals $\mathfrak{m}_1$, …, $\mathfrak{m}_e$, thus it is a semilocal ring with Jacobson radical $\mathfrak{J} := \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_e$. The prime ideals of $\overline{A}$ whose contraction in A coincide with $\mathfrak{m}$ are precisely $\mathfrak{m}_1$, …, $\mathfrak{m}_e$.

For each $i \in \{1,\ldots,e\}$, let us denote by $\overline{\mathfrak{m}}_i$ the extension of $\overline{\mathfrak{m}}$ in the localization $\overline{A}_{\mathfrak{m}_i}$. It is easy to see that

$$\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right) = \prod_{i=1}^{e} \mathrm{G}_{\overline{\mathfrak{m}}_i}\left(\overline{A}_{\mathfrak{m}_i}\right) \ . \tag{1}$$

Proposition 2.1 *If* $\mathrm{Proj}\left(\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)\right)$ *is reduced, then* $\overline{\mathfrak{m}} = \mathfrak{J}$ *and* $\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right) = \mathrm{G}\left(\overline{A}\right)$ *is reduced.*

Proof. Let f be an element in the maximal ideal $\mathfrak{M}_i := \mathfrak{m}_i\overline{A}_{\mathfrak{m}_i}$ of $\overline{A}_{\mathfrak{m}_i}$, and $\overline{f}$ be its coset in $\mathfrak{M}_i/\mathfrak{M}_i^2$. Suppose that

$$\overline{f} \notin \frac{\overline{\mathfrak{m}}_i + \mathfrak{M}_i^2}{\mathfrak{M}_i^2} \ .$$

Since $\overline{A}_{\mathfrak{m}_i}$ is regular, by our standing assumptions on $\overline{A}$, the associated graded ring $\mathrm{G}\left(\overline{A}_{\mathfrak{m}_i}\right)$ is a polynomial ring in $\dim(\overline{A}_{\mathfrak{m}_i})$ indeterminates over the residue field. Therefore, it is not difficult to prove that $f\overline{\mathfrak{m}}_i^n \not\subseteq \overline{\mathfrak{m}}_i^{n+1}$ for all n. This implies that the coset of f in $\overline{A}_{\mathfrak{m}_i}/\overline{\mathfrak{m}}_i = \mathrm{G}_{\overline{\mathfrak{m}}_i}\left(\overline{A}_{\mathfrak{m}_i}\right)_0$ gives a nonzero nilpotent global function on $\mathrm{Proj}\left(\mathrm{G}_{\overline{\mathfrak{m}}_i}\left(\overline{A}_{\mathfrak{m}_i}\right)\right)$.

Then, if Proj $\left(\mathrm{G}_{\overline{\mathfrak{m}}_i}\left(\overline{A}_{\mathfrak{m}_i}\right)\right)$ is reduced, it must be

$$\frac{\mathfrak{M}_i}{\mathfrak{M}_i^2} = \frac{\overline{\mathfrak{m}}_i + \mathfrak{M}_i^2}{\mathfrak{M}_i^2} ,$$

which means $\overline{\mathfrak{m}}_i = \mathfrak{M}_i$, according to the Nakayama Lemma. In particular, $\mathrm{G}_{\overline{\mathfrak{m}}_i}\left(\overline{A}_{\mathfrak{m}_i}\right)$ must be reduced, and taking into account (1), we have that $\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)$ is reduced. The equality $\overline{\mathfrak{m}} = \mathfrak{J}$ also immediately follows. □

Consider now the conductor $\mathfrak{b}$ of A in $\overline{A}$, i.e.,

$$\mathfrak{b} := \left(A : \overline{A}\right) = \{a \in A : \ a\overline{A} \subseteq A\} ,$$

which is an ideal of both A and $\overline{A}$. By $\sqrt{\mathfrak{b}}$ we shall mean the radical in A.

Remark 2.2 *If $\sqrt{\mathfrak{b}} = \mathfrak{m}$, then $\overline{\mathfrak{m}}^n = \mathfrak{m}^n$ for $n >> 0$ and this implies, in turn,*

$$\mathrm{Proj}\left(\mathrm{G}\left(A\right)\right) \cong \mathrm{Proj}\left(\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)\right) .$$

3 The Main Result

For curves, it is not difficult to show that when the projectivized tangent cone at a singular point consists of points in generic position, then the tangent cone is reduced (see Orecchia, 1981, Theorem 3.3). Unfortunately, the proof does not plainly extend to higher dimensions. In this section we give a general theorem which is helpful in this respect. We only need to restrict the hypotheses about the previously considered ring A a bit, by requiring that it is a k-algebra with residue field k. The key point in the proof will be the following general fact.

Let $G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n \trianglerighteq \ldots$ and $G_0' \trianglerighteq G_1' \trianglerighteq \ldots \trianglerighteq G_n' \trianglerighteq \ldots$ be series of subgroups of a group G. Assume that $G_i \subseteq G_i'$, for all i, so that the inclusion maps naturally induce the homomorphisms

$$\nu_i : \frac{G_i}{G_{i+1}} \to \frac{G_i'}{G_{i+1}'} .$$

Consider the following property:

$$\forall G_i,\ \exists G_j' \subseteq G_i . \tag{2}$$

For instance, it certainly holds in the situation of the Artin-Rees lemma.

Lemma 3.1 *Under condition* (2)*, if all the homomorphisms ν_i are surjective, then they are all isomorphisms.*

Proof. Suppose that $\nu_i(\overline{x}) = 0$ for some i. Thus $\overline{x} \in G_i/G_{i+1}$ is the coset of an element $x_{i+1} \in G_{i+1}'$. If $\overline{x}$ is the coset of an element $x_n \in G_n'$, with $n > i$, then we may find a $y \in G_n \subseteq G_n'$ that is in the same G_{n+1}'–coset as x_n, because ν_n is surjective. By composing x_n with the reciprocal of y, we get an element $x_{n+1} \in G_{n+1}'$ that represents $\overline{x}$ as well ($y \in G_n \subseteq G_{i+1}$ since $n > i$). By induction, we deduce that $\overline{x}$ may be represented by an element in G_n', for all $n > i$. According to (2), x may be represented by an element in G_{i+1}, that is, $\overline{x} = 0$. □

Theorem 3.2 *Let A be a reduced, noetherian local k-algebra, with maximal ideal $\mathfrak{m}$ and residue field k, and assume that the normalization $\overline{A}$ is regular and finite over A. Set $Y := \mathrm{Proj}\,(\mathrm{G}\,(A)) \hookrightarrow \mathbb{P}^r$, with $r+1 = \mathrm{emdim}(A)$, and let $\mathfrak{b} = \left(A : \overline{A}\right)$ be the conductor of A in $\overline{A}$. If*

1. *Y is reduced;*
2. *$H(Y,n) = \min\left\{ \binom{n+r}{r}, H\left(\overline{A},n\right) \right\}$;*
3. *the radical $\sqrt{\mathfrak{b}}$ of $\mathfrak{b}$ in A is $\mathfrak{m}$;*

then $\mathrm{G}\,(A)$ is reduced.

Proof. According to Proposition 2.1 and Remark 2.2, the hypotheses 1 and 3 imply that $\mathrm{G}\left(\overline{A}\right)$ is reduced and that the Jacobson radical $\mathfrak{J}$ of $\overline{A}$ coincides with the extension ideal $\mathfrak{m}\overline{A}$.

Let S be the coordinate ring $k[X_0,\ldots,X_r]/I(Y)$ of Y. The canonical epimorphism onto S factors as

$$k[X_0,\ldots,X_r] \twoheadrightarrow \mathrm{G}\,(A) \overset{\varphi}{\twoheadrightarrow} S\ , \tag{3}$$

and $\mathrm{Ker}\,\varphi$ is the saturation of the zero ideal of $\mathrm{G}\,(A)$. Moreover, the natural homomorphism $\nu : \mathrm{G}\,(A) \to \mathrm{G}\left(\overline{A}\right)$ factors as

$$\mathrm{G}\,(A) \overset{\varphi}{\twoheadrightarrow} S \hookrightarrow \mathrm{G}\left(\overline{A}\right)\ , \tag{4}$$

because the hypothesis 3 in the statement implies that ν_n is an isomorphism for $n >> 0$.

Let n_0 be the least degree of homogeneous nonzero elements in $I(Y) \subseteq k[X_0,\ldots,X_r]$ (in the case when $I(Y)=0$ the assert is trivial). For $n < n_0$, looking at (3), we have that φ_n is an isomorphism.

For all $i \geq 0$, set $G_i = \mathfrak{m}^{n_0+i}$ and $G'_i = \mathfrak{J}^{n_0+i}$. Then (2) is satisfied because $\mathfrak{m}^{n_0+i} = \mathfrak{J}^{n_0+i}$ for $i >> 0$. Moreover, the hypothesis 2 in the statement implies that ν_n is surjective for all $n \geq n_0$. Therefore, according to Lemma 3.1, ν_n is an isomorphism for all $n \geq n_0$. Looking at (4), we have that φ_n is an isomorphism for all $n \geq n_0$.

We have shown that φ is an isomorphism. This immediately implies that $\mathrm{G}\,(A)$ is reduced, because it maps isomorphically into a subring of $\mathrm{G}\left(\overline{A}\right)$ (or, alternatively, because it is a graded ring with a saturated zero ideal and a reduced projectivization). □

4 The Conductor of Multiplanar Isolated Surface Singularities

We say that a point P of a (possibly reducible) algebraic variety over k is an *isolated singularity* if it is an isolated point (or, equivalently, it fills an irreducible component) of the singular locus. On the algebraic side, this means that if $A := \mathcal{O}_P$ is the local ring at P, then all localizations of A with respect to its prime ideals are regular, except that at the maximal one. We say that a surface singularity is *multiplanar* if the tangent cone is supported on a union of planes. In this section we state a nontrivial improvement of (Orecchia, 2001, Theorem 3.8]), which gives the conductor for a generic multiplanar isolated singularity (of a surface).

As a preliminary, we recall that a set $l_1, \ldots, l_e$ of distinct lines in $\mathbb{P}_k^r$ is said to be in *generic position* if

$$H(\, l_1 \cup \cdots \cup l_e \, , \, n \,) \;=\; \min \left\{ \binom{n+r}{r}, (n+1)e \right\} \, .$$

Moreover, the same set is said to be in *generic e'-position* (with $e' \leq e$), if every subset made of (precisely) e' lines is in generic position (see Orecchia, 2001, Definition 3.2).

The fact that the word 'generic' is appropriate is nontrivial: it is based on (Hartshorne and Hirschowitz, 1982). This fact is precisely formalized in (Orecchia, 2001, Theorem 3.1), which asserts that when the set of all e-tuples of lines in $\mathbb{P}_k^r$ is parametrized by a suitable open subset $U \subset \mathbb{A}_k^{2(r+1)e}$ in an obvious way, then the subset $U_e \subseteq U$ corresponding to lines in generic position is nonempty and open.

The result we are going to state deals with lines in generic $e-1, e$–position. The subset of U corresponding to such sets of lines is easily obtained by intersecting U_e with the e subsets obtained from U_{e-1} through e suitable projections $\mathbb{A}_k^{2(r+1)e} \to \mathbb{A}_k^{2(r+1)(e-1)}$. Therefore it is open and nonempty as well ([1]).

Note also that lines in generic position are automatically pairwise skew.

Theorem 4.1 *Let A be the local ring at a singular point x of an equidimensional surface over k, such that the normalization $\overline{A}$ is regular; let $\mathfrak{m}$ be the maximal ideal of A, and $\mathfrak{b} := \left(A : \overline{A}\right)$ be the conductor. Assume* $\operatorname{char} k = 0$*, set $r + 1 = \operatorname{emdim}(A)$, and let*

$$n_0 = \min \left\{ n \in \mathbb{N} : (n+1)(e-1) < \binom{n+r}{r} \right\} \, .$$

If

1. $\operatorname{Proj}\left(\operatorname{G}(A)\right) \hookrightarrow \mathbb{P}_k^r$ *is (the reduced subscheme supported on)* $l_1 \cup \cdots \cup l_e$*, with $l_1, \ldots, l_e$ being lines in generic $e-1, e$–position;*
2. *the radical $\sqrt{\mathfrak{b}}$ of $\mathfrak{b}$ in A is $\mathfrak{m}$;*

then

$$\mathfrak{b} = \mathfrak{m}^{n_0} \iff e \neq \left\lfloor \binom{n_0+r}{r} / (n_0+1) \right\rfloor + 1 \, .$$

Proof. The hypothesis 1 states, in particular, that $Y := \operatorname{Proj}\left(\operatorname{G}(A)\right)$ is reduced. Then, taking also into account the hypothesis 2, we immediately deduce from the results of Section 2 that $H\left(\overline{A}, n\right) = (n+1)e$. But the hypothesis 1 also says (in particular) that Y consists of lines in generic position. Hence

$$H(Y,n) = \min \left\{ \binom{n+r}{r} \, , \, (n+1)e \right\} = \min \left\{ \binom{n+r}{r} \, , \, H\left(\overline{A}, n\right) \right\} \, .$$

[1] Of course, the situation would have been similar if we had used a symmetric power of a grassmannian, or a Hilbert scheme, instead of the rough (but handier) parameter space U.

Therefore, Theorem 3.2 assures that $\mathrm{G}(A)$ is reduced. Since $\mathrm{Proj}(\mathrm{G}(A))$ consists of e lines, the cone $\mathrm{Spec}(\mathrm{G}(A))$ consists of e planes. Then, all the hypotheses of (Orecchia, 2001, Theorem 3.8) are verified ([2]), and the result follows from it. $\square$

Remark 4.2 *The condition* $r+1=\mathrm{emdim}(A)$ *in Theorem 4.1 plays an important role. Indeed, even in the case of curves, examples of singularities with a non-reduced tangent cone are known, such that the projectivized tangent cone consists of points in generic position in the space they span.*

5 The Main Example

In this section we construct a wide class of multiplanar isolated surface singularities, for which Theorem 4.1 gives the conductor.

Let $l_1,\dots,l_e$ be pairwise skew lines in $\mathbb{P}^r_k=\mathrm{Proj}(k[X_0,\dots,X_r])$. Assume that each of them does not meet the subspace $X_0=X_1=0$ (it is true for a generic choice of the lines and, when it is not the case, we could perform a coordinate change). Then, for each $i\in\{1,\dots,e\}$, we may choose points $Q_i,Q'_i\in l_i$ with homogeneous coordinates $\mathbf{a}_i=(1,a_{i1},\dots,a_{ir})$ and $\mathbf{b}_i=(0,1,b_{i2},\dots,b_{ir})$, respectively, in such a way that $a_{11},\dots,a_{e1}$ are distinct. Set

$$g(t)=\prod_{i=1}^{e}(t-a_{i1})$$

and let $f_2(t),\dots,f_r(t)$, $h_2(t),\dots,h_r(t)$ be the polynomials of degree $<e$ that are determined by the conditions

$$f_j(a_{i1})=a_{ij},\quad h_j(a_{i1})=b_{ij},\quad i\in\{1,\dots,e\}\ .$$

Consider the ring $R=k[x_0,\dots,x_r]\subseteq k[t,s]$, with

$$\begin{cases} x_0=g\\ x_1=gt+s\\ x_2=gf_2+sh_2\\ \vdots\\ x_r=gf_r+sh_r\end{cases}\quad . \tag{5}$$

Now, $k[t,s]$ is integral over R, since t is integral over $k[x_0]$ and $s=x_1-tx_0$. Then $\Sigma:=\mathrm{Spec}(R)$ is a (rational) surface, naturally embedded in $\mathbb{A}^{r+1}_k=\mathrm{Spec}(k[X_0,\dots,X_r])$ by $X_i\mapsto x_i$. The parametric representation $\mu:\mathbb{A}^2_k\to\Sigma$, given by (5), is finite and dominant, hence surjective (by, e.g., Atiyah and Macdonald, 1969, Proposition 5.10). The origin $O\in\mathbb{A}^{r+1}_k$ is identified with the ideal $\mathfrak{a}=(x_0,\dots,x_n)R\in\Sigma$, and its extension in $k[t,s]$ obviously coincides with $(g,s)=\mathfrak{a}_1\cap\dots\cap\mathfrak{a}_e$, where $\mathfrak{a}_i=(t-a_{i1},s)$. In other words, $\mu^{-1}(O)$ consists of e distinct points $P_1=(a_{11},0)$, …, $P_e=(a_{e1},0)$. Let $A=R_{\mathfrak{a}}$ be the local ring of Σ at O, $\mathfrak{m}=\mathfrak{a}A$ its maximal ideal, and $\overline{A}$ its integral closure in the quotient field $K(A)$ (rings

[2] The multiplicity $e(A)$ is e because the degree of $\mathrm{Proj}(\mathrm{G}(A))$ is e; the point x is non-normal, simply because A is non-regular and $\overline{A}$ is regular.

of fractions will be considered here as subrings of $K(t,s)$). Denote by B the ring of fractions $A\otimes_R k[t,s]=S^{-1}k[t,s]$, with $S=k[t,s]-\bigcup_i \mathfrak{a}_i$ and by $\mathfrak{J}$ its Jacobson radical $(g,s)B=\mathfrak{m}B$. We are interested in calculating the conductor $\mathfrak{b}:=(A:\overline{A})$.

Claim 5.1

1. *$K(A)=k(t,s)$ (i.e, μ is birational) and $\overline{A}=B$;*
2. *the radical $\sqrt{\mathfrak{b}}$ of $\mathfrak{b}$ in A is $\mathfrak{m}$*
3. *$\operatorname{Proj}(\mathrm{G}(A))$ is reduced and equal to $l_1\cup\cdots\cup l_e$*

Writing down the tangent map of μ (represented by $\partial(x_0,\ldots,x_n)/\partial(t,s)$) at P_1, …, P_e, we geometrically see that Σ has e linear branches at O, whose tangent planes project l_1, …, l_e from O. The claim could be deduced from this fact, but we prefer to give a proof which is more elementary from the algebraic viewpoint. The core (still geometric) idea is the following. Let $\alpha=(1,t,f_2,\ldots,f_n)\in k[t]^{n+1}$ and $\beta=(0,1,h_2,\ldots,h_n)\in k[t]^{n+1}$. If, for each $\overline{t}\in k$, $\ell_{\overline{t}}$ denotes the line joining the points in $\mathbb{P}^r_k$ with respective coordinates $\alpha\left(\overline{t}\right)$ and $\beta\left(\overline{t}\right)$ (so that $l_i=\ell_{a_{i1}}$), then the set of lines corresponding to the roots of $g(t)-\lambda$, $\lambda\in k$, are pairwise skew, except for some special values of λ. Then, from

$$k[t,s]^{n+1}\ni(x_0,\ldots,x_n)=g\alpha+s\beta\ , \tag{6}$$

it may be easily deduced that μ is an isomorphism apart from P_1, …, P_e, and the points $\left(\overline{t},\overline{s}\right)$ such that $g(\overline{t})$ takes one of the above mentioned special values. This leads to Claim 5.1. A detailed proof is a matter of applying basic techniques in Algebraic Geometry, from which we now squeeze the algebraic juice, to get a more direct verification.

Consider the ring $C=k[t]\otimes_{k[g]}k[t]$ and, having in mind the identity

$$t^{m+1}\otimes 1\ -\ 1\otimes t^{m+1}\ =\ (t\otimes 1-1\otimes t)\sum_{i=0}^{m}t^{m-i}\otimes t^i\ ,$$

decompose $0=g(t\otimes 1)-g(1\otimes t)$ as $(t\otimes 1-1\otimes t)d$ (thus $\operatorname{Spec}(C/dC)$ is, in ancient terms, the algebraic correspondence determined by g). Denote by $\mathfrak{i}$ the ideal of C generated by d and the fourth order minors of the matrix with columns $\alpha(t\otimes 1),\beta(t\otimes 1),\alpha(1\otimes t),\beta(1\otimes t)$. The obvious ring isomorphism $k[t]/(g)\cong k^e$ naturally induces an isomorphism $C/gC\cong k^{e\times e}$, that with the coset of $p_1\otimes p_2$ associates the e^2 values $p_1(a_{i1})p_2(a_{j1})$. When $i\neq j$, the corresponding value of some of the above minors is nonzero; when $i=j$ the corresponding value of d is $g'(a_{i1})\neq 0$. Then $\mathfrak{i}+gC=C$. But C is finite over $k[g]$, hence $(\mathfrak{i}\cap k[g])+gk[g]=k[g]$ (otherwise, the extension of $(\mathfrak{i}\cap k[g])+gk[g]$ in $k[g]/(\mathfrak{i}\cap k[g])$ would be a proper ideal contained in no proper ideals of the integral extension $C/\mathfrak{i}$ of $k[g]/(\mathfrak{i}\cap k[g])$). Therefore $\mathfrak{i}\cap k[g]$ is generated in $k[g]$ by a nonzero $p=P(g)$, which is invertible in A.

Let us fix a nonzero $f\in\mathfrak{m}$. From (6) it immediately follows that

$$g\alpha(t\otimes 1)+(s\otimes 1)\beta(t\otimes 1)-g\alpha(1\otimes t)-(1\otimes s)\beta(1\otimes t)$$

vanishes in the ring $T:=B_f\otimes_{A_f}B_f$. But $(g,s)B_f=B_f$, hence the natural images of the aforementioned minors vanish in T (it suffices an elementary matrix calculation over T). Since p is invertible in A, this implies that the image of d in T is invertible. Hence $t \otimes 1 - 1 \otimes t$ vanishes in T. Since $B_f = A_f[t]$, we have that T is (naturally isomorphic to) B_f, and this implies that $B_f = A_f$, because B_f is finite over A_f ([3]).

We have $K(A) = K(A_f) = K(B_f) = k(t,s)$. Moreover, $\overline{A} = B$ because B is integrally closed and integral over A. This gives Claim 5.1, 1. The equality $A_f = B_f$ implies that for each $b \in B$, $f^m b \in A$ for a sufficiently large m. Since B is finite over A, m may be chosen as large enough to work for all b. Then $f \in \sqrt{\mathfrak{b}}$. Since f was arbitrarily chosen in $\mathfrak{m} - \{0\}$, this proves the statement 2 of Claim 5.1.

The fact that $A_f = B_f$ for all $f \in \mathfrak{m} - \{0\}$ implies that O is an isolated singularity for Σ, because B is regular. Now, let us consider the tangent cone $\operatorname{Spec}(\mathrm{G}(A))$, which is canonically embedded in $\mathbb{A}_k^{r+1}$ by $X_i \mapsto \overline{x_i}$, where $\overline{x_i}$ denotes the coset of x_i in $\mathfrak{m}/\mathfrak{m}^2 = \mathrm{G}(A)_1$. Since $\sqrt{\mathfrak{b}} = \mathfrak{m}$ and $\mathfrak{J} = \mathfrak{m}\overline{A}$, we have $\operatorname{Proj}(\mathrm{G}(A)) \cong \operatorname{Proj}\left(\mathrm{G}\left(\overline{A}\right)\right)$ by Remark 2.2. But $\mathrm{G}\left(\overline{A}\right) \cong \prod_{i=1}^{e} \mathrm{G}_{\mathfrak{a}_i}(k[t,s])$, and $\mathrm{G}_{\mathfrak{a}_i}(k[t,s])$ is a polynomial ring in two indeterminates $\overline{t - a_{i1}}$, $\overline{s}_i$ over k. Thus $\operatorname{Proj}(\mathrm{G}(A))$ consists of e lines, whose natural embeddings in $\mathbb{P}_k^r$ correspond to the natural homomorphisms

$$k[X_0, \ldots, X_n] \to k\left[\overline{t - a_{i1}}, \overline{s}_i\right], \quad X_j \mapsto \overline{x_j}\ .$$

The points of the line $\operatorname{Proj}\left(k\left[\overline{t - a_{i1}}, \overline{s}_i\right]\right)$ given by the homogeneous prime ideals $(\overline{t - a_{i1}})$ and $(\overline{s}_i)$, are sent through these embeddings into Q_i and Q_i', respectively ([4]). In conclusion, we have

$$\operatorname{Proj}(\mathrm{G}(A)) = l_1 \cup \cdots \cup l_e \tag{7}$$

(when $\operatorname{Proj}(\mathrm{G}(A))$ is considered as a subscheme of $\mathbb{P}_k^r$ in the canonical way). As pointed out in the preliminary discussion in Section 4, the condition 1 in Theorem 4.1 is satisfied for most choices of $l_1, \ldots, l_e$.

Since $\overline{A}$ is obviously regular, if we assume $\operatorname{char} k = 0$ then the only obstruction to applying Theorem 4.1, is the fact that here r has a different

[3]Indeed, tensoring the A_f–module sequence $0 \to A_f \to B_f \to B_f/A_f \to 0$ by B_f we get that $B_f/A_f \otimes B_f = 0$. Since B_f is finite over A_f, if $B_f/A_f \neq 0$, we could choose a submodule M/A_f of B_f/A_f such that B_f/M is generated by exactly one nonzero element. Hence $B_f/M \cong A_f/\mathfrak{q}$ for some proper ideal $\mathfrak{q}$ and $B_f/M \otimes B_f \cong B_f/\mathfrak{q}B_f$ would be 0, which is impossible because B_f is integral over A_f.
An alternative argument, suggested by the geometric insight, may run as follows. It suffices to show that the conductor $\left(A_f : B_f\right)$ is A_f. Suppose the contrary and choose a maximal ideal $\mathfrak{p}$ containing $\left(A_f : B_f\right)$. From $B_f/\mathfrak{p}B_f \otimes_{A_f/\mathfrak{p}} B_f/\mathfrak{p}B_f = B_f/\mathfrak{p}B_f$ follows $B_f/\mathfrak{p}B_f = A_f/\mathfrak{p}$, because $A_f/\mathfrak{p}$ is a field. Then the Nakayama lemma easily implies that the localizations of A_f and B_f at $\mathfrak{p}$ coincide (cf. the proof of Hartshorne, 1977, Chap. II, Lemma 7.4). Hence, for each element b of a finite system of generators of B_f over A_f, we could choose an $a_b \in A_f - \mathfrak{p}$ such that $a_b b \in A_f$. Then the product of the a_b's would be an element in $\left(A_f : B_f\right)$ lying outside $\mathfrak{p}$, and this contradicts the choice of $\mathfrak{p}$.

[4]Look at the congruences

$$x_0 \equiv \rho_i(t - a_{i1}),\ x_1 \equiv \rho_i a_{i1}(t - a_{i1}), \ldots,\ x_j \equiv \rho_i f_j(a_{i1})(t - a_{i1}), \ldots$$

mod. $\mathfrak{a}_i^2 + (s)$ in $k[t,s]$, where $\rho_i = \prod_{j \neq i}(a_{i1} - a_{j1}) = g'(a_{i1})$, and

$$x_0 \equiv 0,\ x_1 \equiv s, \ldots,\ x_j \equiv h_j(a_{i1})s, \ldots$$

mod. $\mathfrak{a}_i^2 + (t - a_{i1})$.

meaning (cfr. Remark 4.2). But from the definition of generic position it immediately follows that, in our example, $r+1 = \mathrm{emdim}(A)$ if e is sufficiently large (namely, $e \geq (r+1)/2$). Moreover, even if e is small, a simple change of coordinates shows that the surface Σ is contained in the subspace of $\mathbb{P}^r_k$ spanned by the lines. Thus, we may replace $\mathbb{P}^r_k$ with this subspace.

In conclusion, when $\mathrm{char}\, k = 0$, Theorem 4.1 applies to the example constructed in this section for most of the choices of the lines. Hence we get $\mathfrak{b} = \mathfrak{m}^{n_0}$, with $n_0 = \min\left\{n \in \mathbb{N} : (n+1)(e-1) < \binom{n+r}{r}\right\}$, provided that

$$e \neq \left\lfloor \binom{n_0+r}{r} / (n_0+1) \right\rfloor + 1 \, .$$

But this numeric condition is verified in most cases, as pointed out in (Orecchia, 2001, Remark 3.5). Therefore, for a wide class of isolated surface singularities the conductor is definitely found.

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